Uniqueness Theorem for Stationary Black Hole Solutions of σ-models in Five-dimensions

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We prove the uniqueness theorem for stationary self-gravitating non-linear σ-models in five-dimensional spacetime. We show that the Myers-Perry vacuum Kerr spacetime is the only maximally extended, stationary, axisymmetric asymptotically flat solution having the regular rotating event horizon with a constant mapping.

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I. INTRODUCTION

Recently interests in higher dimensional black holes have renewed. The unification attempts such as M/string theory described our Universe as a brane or defect emerged in higher dimensional geometry. $E8 \times E8$ heterotic string theory at strong coupling is described in terms of M-theory acting in eleven-dimensional spacetime with boundaries where ten-dimensional Yang-Mills gauge theories reside on two boundaries [1]. The so-called TeV gravity attracts attention to higher dimensional black hole because of the suggestion that such kind of objects may be produced in the near future, in high energy experiments [2]. This kind of black holes are classical solutions of higher dimensional Einstein's equations. The radius of their event horizon is much smaller that the scale of extra dimensions.

The classification of non-singular black hole solutions began with the Israel's work [3], then Müller zum Hagen *et al.* [4] and Robinson [5], provided other contributions to the problem [6–10]. In Refs. [11,12] both for vacuum and Einstein-Maxwell (EM) black holes the condition of non-degeneracy of the event horizon was removed. It was shown that for the static electro-vacuum black holes all degenerate components of the event horizon should have charges of the same signs.

On the other hand the problem of uniqueness theorem for stationary axisymmetric black hole was considered in Refs. [13,14] and the complete proof was delivered by Mazur [15] and Bunting [16] (see also for a review of the uniqueness of black hole solutions story see [17] and references therein).

Studies of the low-energy string theory also triggers the resurgence of works concerning the mathematical aspects of the black holes emerging in it. Namely, the staticity theorem for Einstein-Maxwell axion dilaton (EMAD) gravity was studied in Ref. [18] and uniqueness of the black hole solutions in dilaton gravity was proved in works [19,20], while the uniqueness of the static dilaton $U(1)^2$ black holes being the solution of $N = 4, d = 4$ supergravity was provided in [21]. The extension of the proof to $U(1)^N$ static dilaton black holes was established in Ref. [22].

The possibility of production of higher dimensional black holes in accelerators caused the considerable interests in n-dimensional black hole uniqueness theorem, both in vacuum and charged case [23–26]. The complete classification of n-dimensional charged black holes having both degenerate and non-degenerate components of event horizon was provided

in Ref. [27]. Uniqueness theorem for n-dimensional static black hole carrying *electric* and *magnetic* components of $(n-2)$-gauge form was proved in [28].

The primary signature of appearing black holes in future accelerator experiments will be a Hawking emission and generically the black holes will have angular momenta. Therefore the uniqueness of rotating black holes in higher dimensions is of a great importance in studies mathematical properties of them. However the problem of uniqueness theorem for stationary n-dimensional black holes is much more complicated. It was shown that generalization of Kerr metric to arbitrary n-dimensions proposed by Myers-Perry [29] is not unique (see for the counterexample showing that a five-dimensional rotating black hole ring solution has the same angular momentum and mass but its horizon is homeomorphic to $S^2 \times S^1$ Refs. [30] and [31]). It has been proved that Myers-Perry solution is the unique black hole in five-dimensions in the class of spherical topology and three commuting Killing vectors [32]. The conditions for uniqueness of black holes or black rings in five-dimensional gravity was discussed in Ref. [33].

The uniqueness theorem for self-gravitating nonlinear σ-models in higher dimensional spacetime was obtained in [34]. Recently, the related issues for supersymmetric black holes were given in Refs. [35].

In this paper we shall treat the problem of uniqueness of five-dimensional axisymmetric, stationary self-gravitating σ-models. The main result established in our work is that the only regular black hole solution with regular rotating event horizon is the five-dimensional vacuum Kerr black hole with constant mapping.

II. FIVE-DIMENSIONAL ROTATING σ-MODELS

In our paper we shall consider the action describing the n-dimensional self-gravitating σ-model written as

$$I = \int d^n x \, \sqrt{-{}^{(n)}g} \bigg[{}^{(n)}R - \frac{1}{2} G_{AB}(\varphi(x)) \; \varphi^A_{,\mu} \varphi^{B,\mu} \bigg]. \tag{1}$$

It then follows directly that the energy momentum for the model has the form

$$T_{\mu\nu}(\varphi) = G_{AB}(\varphi(x)) \; \varphi^A_{,\mu} \varphi^B_{,\nu} - \frac{1}{2} G_{AB}(\varphi(x)) \; \varphi^A_{,\gamma} \varphi^{B,\gamma} g_{\mu\nu}. \tag{2}$$

The equations of motion derived from the variational principle are as

$$\nabla_\gamma \nabla^\gamma \varphi^A + \Gamma^A_{BC} \; \varphi^B_{,\mu} \varphi^{C,\mu} = 0, \tag{3}$$

$${}^{(n)}G_{\mu\nu} = T_{\mu\nu}(\varphi). \tag{4}$$

In what follows we shall take into account the asymptotically, five-dimensional flat spacetime, i.e., the spacetime will contain a data set $(\Sigma_{end}, g_{ij}, K_{ij})$ with scalar fields of φ such that a spacelike hypersurface Σ_{end} is diffeomorphic to $\mathbf{R^4}$ minus a ball. The asymptotical conditions of the following forms should also be satisfied:

$$\mid g_{ij} - \delta_{ij} \mid + r \mid \partial_a g_{ij} \mid + \ldots + r^m \mid \partial_{a_1 \ldots a_m} g_{ij} \mid + r \mid K_{ij} \mid + \ldots + r^m \mid \partial_{a_1 \ldots a_{m-1}} K_{ij} \mid \leq \mathcal{O}\bigg(\frac{1}{r}\bigg), \tag{5}$$

where g_{ij} and K_{ij} are induced on Σ_{end}. K_{ij} is the extrinsic curvature tensor of the hypersurface Σ_{end}. It is required that in the local coordinates on Σ_{end} the scalar field satisfies the following fall-off condition:

$$\varphi^A = \varphi^A_\infty + \mathcal{O}\bigg(\frac{1}{r^{3/2}}\bigg), \tag{6}$$

Our spacetime will admit three commutating Killing vector fields $k_\mu, \phi_\mu, \psi_\mu$

$$[k, \phi] = [k, \psi] = [\phi, \psi] = 0, \tag{7}$$

where k_μ is an asymptotically timelike for which $V = -k_\mu k^\mu$, while vectors ϕ_μ and ψ_μ are spacelike. They all have closed orbits and the following is satisfied:

$$\mathcal{L}_k \ g_{\mu\nu} = \mathcal{L}_\phi \ g_{\mu\nu} = \mathcal{L}_\psi \ g_{\mu\nu} = 0, \tag{8}$$

where $\mathcal{L}$ stands for the Lie derivative with respect to the adequate Killing vector fields. We shall also assume that the scalar field φ is invariant due to the action of Killing vector fields, namely

$$\mathcal{L}_k \ \varphi = \mathcal{L}_\phi \ \varphi = \mathcal{L}_\psi \ \varphi = 0. \tag{9}$$

The metric in the spacetime under consideration can be written in the Weyl-Papapetrou form as follows:

$$ds^2 = -\frac{\rho^2}{f} dt^2 + f_{ab} \Big(dx^a + \omega^a dt \Big) \Big(dx^b + \omega^b dt \Big) + \frac{e^{2\sigma}}{f} \Big(d\rho^2 + dz^2 \Big), \tag{10}$$

where all functions appearing in the above metric have the only ρ and z dependence. Furthermore, the metric (10) can also be rearrange as

$$ds^2 = \sigma_{ab} dx^a dx^b + \gamma_{ij} dx^i dx^j, \tag{11}$$

where $a, b = t, \phi, \psi$ and comprises the first two components in expression (10) while $i, j = \rho, z$ and describes $\gamma_{ij} = X g_{ij}$. g_{ij} stands for the metric of a flat spacetime written in (ρ, z) coordinates. The conformal factor is equal to $X = e^{2\sigma}/f$. Using rules of conformal transformation, after some algebra, we find expressions for the Ricci tensor components:

$$\begin{aligned} R_{ij} = {}^{(\gamma)}R_{ij} + \frac{1}{2}\gamma_{ij} {}^{(\gamma)}\nabla^2 \ln X - \frac{1}{2\rho X} \Big({}^{(\gamma)}\nabla_i \rho \ {}^{(\gamma)}\nabla_j X + {}^{(\gamma)}\nabla_i X \ {}^{(\gamma)}\nabla_j \rho - \gamma_{ij} {}^{(\gamma)}\nabla^k \rho \ {}^{(\gamma)}\nabla_k X \Big) \\ - \frac{1}{\rho} {}^{(\gamma)}\nabla_i {}^{(\gamma)}\nabla_j \rho + \frac{1}{\rho^2} {}^{(\gamma)}\nabla_i \rho \ {}^{(\gamma)}\nabla_j \rho + \frac{1}{4} {}^{(\gamma)}\nabla_i \sigma^{ab} \ {}^{(\gamma)}\nabla_j \sigma_{ab}. \end{aligned} \tag{12}$$

Consequently equations of motion yield

$$R_{\rho\rho} - R_{zz} + \frac{1}{\rho X} {}^{(g)}\nabla_\rho X - \frac{1}{\rho^2} + \frac{1}{4} \Big({}^{(g)}\nabla_z \sigma^{ab (g)}\nabla_z \sigma_{ab} - {}^{(g)}\nabla_\rho \sigma^{ab (g)}\nabla_\rho \sigma_{ab} \Big) = \frac{2}{\rho} \ {}^{(g)}\nabla_\rho \sigma, \tag{13}$$

$$2R_{z\rho} + \frac{1}{\rho X} {}^{(g)}\nabla_z X - {}^{(g)}\nabla_z \sigma^{ab (g)}\nabla_\rho \sigma_{ab} = \frac{2}{\rho} \ {}^{(g)}\nabla_z \sigma, \tag{14}$$

$$\begin{aligned} R_{zz} + R_{\rho\rho} - {}^{(g)}\nabla_m {}^{(g)}\nabla^m \ln X + \frac{1}{\rho X} {}^{(g)}\nabla_\rho X - \frac{1}{\rho X} {}^{(g)}\nabla^j \rho {}^{(g)}\nabla_j X - \frac{1}{\rho^2} + \\ - \frac{1}{4} \Big({}^{(g)}\nabla_\rho \sigma^{ab} \ {}^{(g)}\nabla_\rho \sigma_{ab} + {}^{(g)}\nabla_z \sigma^{ab} \ {}^{(g)}\nabla_z \sigma_{ab} \Big) = -2 {}^{(g)}\nabla^j {}^{(g)}\nabla_j \sigma, \end{aligned} \tag{15}$$

where ${}^{(g)}\nabla$ is the derivative with respect to g_{ij} metric. From the above implies directly that $\sigma(\rho, z)$ reduces to the sum of two components, i.e.,

$$\sigma = \sigma(vac) + \sigma(\varphi), \tag{16}$$

where $\sigma(vac)$ is the solution of five dimensional vacuum equations of motion while $\sigma(\varphi)$ is connected with the solution of matter equations.

The equations of motion for self-gravitating non-linear σ-model yield

$$\frac{1}{\rho}{}^{(g)}\nabla_z\sigma(\varphi) = \frac{1}{2}G_{AB}(\varphi(x))\Big({}^{(g)}\nabla_\rho\varphi^A\ {}^{(g)}\nabla_z\varphi^B + {}^{(g)}\nabla_z\varphi^A\ {}^{(g)}\nabla_\rho\varphi^B\Big), \tag{17}$$

$$\frac{1}{\rho}{}^{(g)}\nabla_\rho\sigma(\varphi) = \frac{1}{2}G_{AB}(\varphi(x))\Big({}^{(g)}\nabla_\rho\varphi^A\ {}^{(g)}\nabla_\rho\varphi^B - {}^{(g)}\nabla_z\varphi^A\ {}^{(g)}\nabla_z\varphi^B\Big), \tag{18}$$

$${}^{(g)}\nabla_m{}^{(g)}\nabla^m\sigma(\varphi) = -\frac{1}{2}G_{AB}(\varphi(x))\Big({}^{(g)}\nabla_\rho\varphi^{A(g)}\nabla_\rho\varphi^B + {}^{(g)}\nabla_z\varphi^{A(g)}\nabla_z\varphi^B\Big). \tag{19}$$

In order to prove the uniqueness theorem for five-dimensional σ-model we shall use the strategy presented in Ref. [36]. Namely, we choose a two-dimensional vector

$$\Pi_j = \rho\ {}^{(g)}\nabla_j e^{-\sigma(\varphi)}. \tag{20}$$

Henceforth, by virtue of Stokes' theorem for Π_j vector and integration over the domain of outer communication $<< \mathcal{J} >>$ we find the following:

$$\begin{aligned} D_{\partial\mathcal{J}} &= \int_{\partial\mathcal{J}} \rho e^{-\sigma(\varphi)}\Big({}^{(g)}\nabla_z\sigma(\varphi)\ d\rho - {}^{(g)}\nabla_\rho\sigma(\varphi)\ dz\Big) \\ &= \int_{<<\mathcal{J}>>} d\rho dz\ \rho e^{-\sigma(\varphi)}\Big[{}^{(g)}\nabla^i\sigma(\varphi){}^{(g)}\nabla_i\sigma(\varphi) - \Big(\frac{1}{\rho}{}^{(g)}\nabla_\rho\sigma(\varphi) + {}^{(g)}\nabla^{i(g)}\nabla_i\sigma(\varphi)\Big)\Big]. \end{aligned} \tag{21}$$

From Eqs.(17),(18) and (19) it follows in particular that the second term on the right-hand side of (21) is greater or equal to zero. It implies that the right-hand side is the sum of two non-negative terms.

Now, let us calculate the left-hand side of expression (21). In order to do so we introduce an ellipsoidal type of coordinates [13] given as:

$$\rho^2 = (\lambda^2 - c^2)(1-\mu^2), \qquad z = \lambda\mu, \tag{22}$$

where $\mu = \cos 2\theta$ is chosen in such a way that the event horizon boundary occurs for a constant value of $\lambda = c$. Two rotation axis segments distinguishing the north and south of the horizon are given by the respective limit $\mu = \pm 1$. We take into account the domain of outer communication $<< \mathcal{J} >>$ as a rectangle, i.e.,

$$\begin{aligned} \partial\mathcal{J}^{(1)} &= \{\mu = 1,\ \lambda = c, \dots, R\}, \\ \partial\mathcal{J}^{(2)} &= \{\lambda = c,\ \mu = 1, \dots, -1\}, \\ \partial\mathcal{J}^{(3)} &= \{\mu = -1,\ \lambda = c, \dots, R\}, \\ \partial\mathcal{J}^{(4)} &= \{\lambda = R,\ \mu = -1, \dots, 1\}. \end{aligned} \tag{23}$$

Next one rewrites the left-hand side of Eq.(21) in ellipsoidal coordinates as functions of λ and μ

$$\begin{aligned} D_{\partial\mathcal{J}} &= \int_{\partial\mathcal{J}} \rho e^{-\sigma(\varphi)}\Big({}^{(g)}\nabla_z\sigma(\varphi)\ d\rho - {}^{(g)}\nabla_\rho\sigma(\varphi)\ dz\Big) = \\ &= \int_{\partial\mathcal{J}} e^{-\sigma(\varphi)}\Big[(1-\mu^2)\sigma(\varphi)_{,\mu}\ d\lambda - (\lambda^2 - c^2)\sigma(\varphi)_{,\lambda}\ d\mu\Big], \end{aligned} \tag{24}$$

where using Eqs.(17) and (18) in ellipsoidal type of coordinates one reaches to the following expressions for $\sigma(\varphi)_{,\lambda}$ and $\sigma(\varphi)_{,\mu}$:

$$\sigma(\varphi)_{,\lambda} = \frac{G_{AB}(1-\mu^2)}{2(\lambda^2-\mu^2c^2)}\bigg[-\lambda(1-\mu^2)\ \varphi^A_{,\mu}\varphi^B_{,\mu} + \lambda(\lambda^2-c^2)\ \varphi^A_{,\lambda}\varphi^B_{,\lambda} - 2\mu(\lambda^2-c^2)\ \varphi^A_{,\lambda}\varphi^B_{,\mu}\bigg], \tag{25}$$

$$\sigma(\varphi)_{,\mu} = \frac{G_{AB}(\lambda^2-c^2)}{2(\lambda^2-\mu^2c^2)}\bigg[-\mu(1-\mu^2)\ \varphi^A_{,\mu}\varphi^B_{,\mu} + \mu(\lambda^2-c^2)\ \varphi^A_{,\lambda}\varphi^B_{,\lambda} + 2\lambda(1-\mu^2)\ \varphi^A_{,\lambda}\varphi^B_{,\mu}\bigg]. \tag{26}$$

Having in mind relations (25) and (26) we can establish that $\sigma(\varphi)_{,\lambda}, \sigma(\varphi)_{,\mu}$ and $e^{-\sigma(\varphi)}$ remain finite along the boundaries $\partial\mathcal{J}^{(i)}$ for $i = 1, 2, 3$ and they all vanish along these parts of the boundary. It remains to consider the last part of the boundary. In order to do so one uses the formula for λ in terms of radial coordinates r [32], as follows:

$$\lambda = \frac{r^2}{2} + \frac{a^2+b^2}{4} + \mathcal{O}\bigg(\frac{1}{r}\bigg), \tag{27}$$

where the parameters a and b are bounded with two independent angular momentum of five-dimensional Kerr black hole, i.e., $J_\phi = \pi m a/4G$ and $J_\psi = \pi m b/4G$. The mass of black hole is connected with m parameter by the relation $M = 3\pi m/8G$. Hence, it implies

$$D_{\partial\mathcal{J}^{(4)}} = -\lim_{R\to\infty}\int_{D_{\partial\mathcal{J}^{(4)}}} e^{-\sigma(\varphi)}(\lambda^2-c^2)\ \sigma_{,\lambda}\ d\mu = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \lim_{r\to\infty}\bigg(e^{-\sigma(\varphi)}r^3\sigma_{,r}\ \sin 2\theta\ d\theta + \mathcal{O}\bigg(\frac{1}{r^n}\bigg)\bigg), \tag{28}$$

where $n \geq 7$. Having in mind the asymptotical properties of the derivatives of scalar field φ

$$\varphi^A_{,\theta} = \mathcal{O}\bigg(\frac{1}{r^{3/2}}\bigg), \qquad \varphi^A_{,r} = \mathcal{O}\bigg(\frac{1}{r^{5/2}}\bigg), \tag{29}$$

we conclude that the above entire integral vanishes to the fact that $\lim_{r\to\infty} r^3\ \sigma_{,r} = 0$.
Hence ${}^{(g)}\nabla^i\sigma(\varphi){}^{(g)}\nabla_i\sigma(\varphi)$ and $\frac{1}{\rho}{}^{(g)}\nabla_\rho\sigma(\varphi) + {}^{(g)}\nabla^{i}{}^{(g)}\nabla_i\sigma(\varphi)$ are equal to zero. It occurs that $\sigma(\varphi)$ is constant in the considered domain of outer communication, but using the fact that $\sigma(\varphi)$ tends to zero as $r \to \infty$ we get that $\sigma(\varphi) = 0$ which in turn implies that φ is constant in the entire domain $<< \mathcal{J} >>$. Just from Eq.(16) we have obtained the only $\sigma(vac)$ solution of equations of motion. In Ref. [32] uniqueness of the asymptotically flat, stationary five-dimensional black hole solution being the solution of Einstein vacuum equations with regular event horizon homeomorphic to S^3 and admitting three commutating Killing vector fields (two spacelike and one timelike) was presented. Thus, we can assert the main conclusion of our work:

Theorem:

Let us consider a stationary axisymmetric solution to five-dimensional self-gravitating non-linear σ-models with an asymptotically timelike Killing vector field k_μ and two spacelike Killing vector fields ϕ_μ and ψ_μ. The scalar field is invariant under the action of the Killing vector fields. Then, the only black hole solution with regular rotating event horizon in the asymptotically flat, strictly stationary domain of outer communication is the five-dimensional Myers-Perry vacuum Kerr black hole solution with a constant mapping φ.

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